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Relation between certain standard representations of $\underline{g}$ and non-unitary principal series representations of G. Case of $G = SO_0(n-1,1)$, $\underline{g} = \underline{so}(n-1,1)$.

Here we explain how we can establish an explicit correspondence between these two kinds of representations, and then can study the structure of the latter representations (subquotient phenomena). We keep to the notations in the following papers.

[I] On infinitesimal operators of irreducible representations of the Lorentz group of n-th order, Proc. Japan Acad., 38(1962), 83-87.

[II] On irreducible representations of the Lorentz group of n-th order, ibid., 38(1962), 258-262.

[III] The characters of irreducible representations of the Lorentz group of n-th order, ibid., 41(1965), 526-531.

Errata to [II]. p.259, line 5↑ (from bottom), read "$0 < p \leqslant n_1$" instead of "$0 < p < n_1$"; p.260, line 3↑ read "where exactly (j-1) number of n_i's are zero in α ($j \geqslant 1$) and"; p.261, line 8↓ read "where exactly j number of n_i's are zero in α ($j \geqslant 1$) and"; p.262, line 2↓, at the end of this line, add "$\rho \neq 0$".

At the stage when these papers were written, any proof could not be found for the Gelfand-Cejtlin formula of the infinitesimal operators for SO(n) in [1]. Therefore we gave a proof of it at the same time as to solve our present question, using a kind of zig-zag induction on n such as $\underline{so}(n) \longrightarrow \underline{so}(n,1) \longrightarrow \underline{so}(n+1) \longrightarrow \underline{so}(n+1,1) \ldots$

To clarify what are known and what must be proved, we summalize the known facts and pose some questions. For simplicity, we restrict ourselves to the case of one-valued representations of SO(n) and $SO_0(n-1,1)$. (The spin case can be treated likewise.)

[A] Finite-dimensional representations of $SO(n)$ (or those of $SO_0(n-1,1)$ are parametrized by highest weights.

(A1) The Weyl's character formula is given by means of the highest weights.

(A2) Hence we know their infinitesimal characters in the sense in [III, §4].

(A3) We know how an irreducible representation of $SO(n)$ splits when it is restricted to $\begin{pmatrix} SO(n-1) & \\ & 1 \end{pmatrix}$. This splitting diagram determines uniquely the original representation of $SO(n)$.

Question (QAn). Prove the Gelfand-Cejtlin formula for $SO(n)$, that is, prove that the infinitesimal operators $A_{i,i-1}$'s given by the formulas in [I] satisfy the commutation relations in [I,(5)].

Question (QAIn). Determine the highest weight of the representation given by the operators of Gelfand-Cejtlin. Then determine its infinitesimal character.

Lemma 1. (QAI(n-1)) + (QAn) $\Longrightarrow$ (QAIn).

Proof. This follows from (A2) and (A3).

[B] Non-unitary principal series. Put

$$K = \begin{pmatrix} SO(n-1) & \\ & 1 \end{pmatrix}, \quad M = \begin{pmatrix} SO(n-2) & \\ & 1_2 \end{pmatrix}, \quad A = \{a(t) = g_{n-1}(t);\ t \in \mathbb{R}\},$$

$$N = \left\{ \begin{pmatrix} 1_{n-2} & -{}^t\xi & {}^t\xi \\ \xi & 1-|\xi|^2/2 & |\xi|^2/2 \\ \xi & -|\xi|^2/2 & 1+|\xi|^2/2 \end{pmatrix};\ \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n-2} \end{pmatrix} \in \mathbb{R}^{n-2},\ |\xi|^2 = \sum_i \xi_i^2 \right\}.$$

For $g = na(t)u$ ($n \in N$, $a(t) \in A$, $u \in K$), we put $\alpha_c(g) = \exp(c+(n-1)/2)t$ for $c \in \mathbf{C}$. For $u \in K$, $g \in G$, define $u\bar{g} \in K$ as $ug = n'a'u\bar{g}$ with $n' \in N$, $a' \in A$. On $L^2(K)$, we define a unitary representation of $SO_0(n-1,1)$: $g \mapsto T_c(g)$ by

$$(T_c(g)f)(u) = \alpha_c(ug)\, f(u\bar{g}) \quad (f \in L^2(K)).$$

Let α be the parameter in [I, II] of an irreducible representation π_α of M. By the projection P_α corresponding to π_α, we get a canonical realization $T_{\alpha,c}$ on a space H_α of the induced representation of (α, c).

(B1) We have the character formula for $T_{\alpha,c}$.

(B2) Hence we know the infinitesimal character of $T_{\alpha,c}$.

(B3) We know for which values of c, $T_{\alpha,c}$ contains a non-trivial finite-dimensional invariant subspace.

(B4) Let δ_μ be an irreducible representation of K with highest weight μ. Then δ_μ is contained in $T_{\alpha,c}|K$ (with multiplicity 1) if and only if $\delta_\mu|M \supset \pi_\alpha$. (This relation is denoted by $\mu \supset \alpha$.) Independently of c, we fix a canonical orthonormal basis in the subspace $H_\alpha(\delta_\mu)$ of H_α corresponding to δ_μ in such a way that the infinitesimal operators for K are given by the Gelfand-Cejtlin formula with respect to this basis. Then the infinitesimal operator A corresponding to $\{a(t);\ t \in \mathbb{R}\}$ is expressed with respect to this basis of H_α in such a way that every entry is linear in c. ($A = \frac{d}{dt} T_{\alpha,c}(a(t))\big|_{t=0}$.)

[C] <u>A standard representation</u> $S_{\alpha,c}$ of $\underline{g}$. Fix $c \in \mathbb{C}$. We consider a vector $\xi'(\lambda)$ corresponding to every $\xi(\lambda)$ in [I]. The operators $A'_{i,i-1}$'s are given by the same formulas in [I] as for $A_{i,i-1}$'s respectively. The new operator B'_{n-1} corresponding to $g_{n-1}(t) = a(t)$ is given by a slite modification of B_{n-1} in [I] as follows. Consider the case $n = 2k+1$. (The case

$n = 2k+2$ can be treated analogously.)

$$B'_{n-1}\xi'(\lambda) = \sum_{j=1}^{k} A'^{j}_{-}(\lambda)\,\xi'(\lambda^{j+}_{2k-1}) + \sum_{j=1}^{k} A'^{j}_{+}(\lambda)\,\xi'(\lambda^{j-}_{2k-1})$$

where $A'^{j}_{\pm}(\lambda)$ are obtained from $A^{j}(\lambda)$ in [I,(15)] by replacing $(c^2-(\ell_{2k-1}+1/2)^2)^{1/2}$ in the numerator by $(\ell_{2k-1}+1/2)\pm c$ respectively and multiplying them by $i = \sqrt{-1}$.

Question (QCn). For an α, let λ run over all possible tables corresponding to all $\mu \supset \alpha$. Then, prove that the above operators $A'_{i,i-1}$ and B'_{n-1} give a representation of $\underline{g}$ (i.e., prove the commutation relations holds). (We denote this representation by $S_{\alpha,c}$.)

(C1) In the formulas (13)-(15) in [I], we make a certain base-change such that

(*) $\xi'(\lambda)$ = (a rational function in c) $\times\ \xi(\lambda)$,

then we get the above new operators $A'_{i,i-1}$ and B'_{n-1}. Let Z_α be the set of half-integers different from $\pm\ell_1, \pm\ell_2, \ldots, \pm\ell_{k-1}$. Then outside of Z_α the rational functions in c appearing in (*) have no poles and no zeros, and therefore the representation of $\underline{g}$ given by $A'_{i,i-1}, B'_{n-1}$ is equivalent to that given by $A_{i,i-1}, B_{n-1}$. (For an exceptional value of c, the former is usually not completely reducible, contrary to the latter.)

(C2) Since the representation $S_{\alpha,c}$ is algebraically irreducible for $c \notin Z_\alpha$, and it is $\underline{k}$-finite ($\underline{k}$ = the Lie algebra of K), we see that for any element Z in the center $\underline{Z}$ of the envelopping algebra $U(\underline{g}_c)$ acts as a scalar multiplication. For any $Z \in \underline{Z}$, all the entries in $S_{\alpha,c}(Z)$ are polynomials in c. Therefore it is a scalar operator for any $c \in \mathbb{C}$.

Question (QCIn). Determine the infinitesimal character $\underset{\sim}{Z} \to \mathbb{C}$ of $S_{\alpha,c}$.

(C3) From the explicit form of B'_{n-1}, we see that for $c \in Z_\alpha$, $c > \ell_{k-1}$, $S_{\alpha,c}$ has a finite-dimensional invariant subspace. The representation of $\underset{\sim}{g}$ on this invariant subspace corresponds to the representation of $\underset{\sim}{so}(n)$ given by Gelfand-Cejtlin with the parameter $(\alpha, c-(k-1/2))$.

(C4) For $c < 0$, there exists no finite-dimensional invariant subspace for $S_{\alpha,c}$.

Problem (Q). Establish an explicit correspondence between $T_{\alpha,c}$'s and $S_{\alpha,c}$'s.

Lemma 2. (QCn) $\Longrightarrow$ (QAn).

Proof. This follows from (C3).

Lemma 3. (QCn) + (QA(n-1)) $\Longrightarrow$ (QCIn).

Proof. Note that the scalar $S_{\alpha,c}(Z)$ for $Z \in \underset{\sim}{Z}$ is a polynomial in c, and therefore it is uniquely determined by the values for $c \in Z_\alpha$, $c > \ell_{k-1}$. Then the assertion follows from Lemmas 1, 2 and (C3). Q.E.D.

Lemma 4. (QA(n-1)) $\Longrightarrow$ (QCn).

We give an outline of a proof. As is shown in [I], it is sufficient to prove the commutation relations for B'_{n-1} corresponding to (6) in [I]. This is essentially reduced to prove some identities of polynomials such as $P_\nu = 0$ $(0 \leqslant \nu \leqslant n-1)$, $P_{n-1} = 1$

in [2, p.145, (5)] (cf. [3]).

Thus using Lemmas 1-4, we can solve (QAn), (QAIn), (QCn) and (QCIn) by induction on n.

To solve (Q), we proceed as follows. We now look at the imaginary axis of $\mathbb{C}$, that is, we compair $S_{\alpha,i\rho}$ with $T'_{\alpha,i\rho}$ = the representation of $\underset{\sim}{g}$ corresponding to $T_{\alpha,i\rho}$ ($\rho \in \mathbb{R}$, $\rho \neq 0$). Note that $S_{\alpha,i\rho}(X)$ is skew-hermitian for $X \in \underset{\sim}{g}$, and that so is $T'_{\alpha,i\rho}(X)$ because $T_{\alpha,i\rho}$ is unitary. They are both algebraically irreducible. We see from the subquotient theorem of Harish-Chandra that $S_{\alpha,i\rho}$ is isomorphic to a subquotient representation of some $T'_{\alpha',c'}$. Compairing the infinitesimal characters of them (by (QCIn) and (B2)), we see that $S_{\alpha,i\rho}$ must be algebraically equivalent to $T'_{\alpha,i\rho}$ (and to $T'_{\alpha,-i\rho}$ when $n = 2k+1$ for exemple). Let $R(i\rho)$ be an intertwining operator between them:

(**) $\quad S_{\alpha,i\rho}(X)\, R(i\rho) = R(i\rho)\, T'_{\alpha,i\rho}(X) \qquad (X \in \underset{\sim}{g}).$

Then by the normalization of a basis in every $H_\alpha(\delta_\mu)$, there exist constants $d_\mu(i\rho)$ adjusting the bases in $H_\alpha(\delta_\mu)$'s, such that

$$R(i\rho) = \sum_{\mu \supset \alpha} d_\mu(i\rho)\, P^\mu ,$$

where P^μ denotes the projection onto $H_\alpha(\delta_\mu)$.

By the explicit form of B'_{n-1} for $S_{\alpha,i\rho}$, and by (B4), we see that $d_\mu(i\rho)$ must be a rational function in $i\rho$ essentially. Therefore $R(c)$ can be defined and the intertwinig relation (**) holds also for almost all $c \in \mathbb{C}$: $S_{\alpha,c}(X)\, R(c) = R(c)\, T'_{\alpha,c}(X)$. Compairing the places of c where finite demensional representations split out, we see that $R(c)$ must be a multiple of an operator independent of c. This proves that $S_{\alpha,c}$ and $T'_{\alpha,c}$ are equivalent to each other and we can take the adjusting constants $d_\mu(c)$ independent of c.

Thus the problem (Q) is now completely solved.

Once this correspondence has been established, we can study the subquotient structure of the representations $T_{\alpha,c}$ from the infinitesimal standpoint, that is, using only the standard representation $S_{\alpha,c}$ of $\underset{\sim}{g}$.

References

[1] I. M. Gelfand and M. L. Cejtlin: The finite-dimensional representations of the groups of orthogonal matrices (in Russian), DAN SSSR, 71, 1017-1020(1950).

[2] A. Weil: Basic number theory, Springer-Verlag, Berlin, 1967.

[3] D. P. Zhelobenko: Compact Lie groups and their representations (in Russian), Moskow, 1970.

Remark added. The representations $T_{\alpha',c'}$ with the same infinitesimal character as for $S_{\alpha,c}$ are $T_{\alpha,c}$ and $T_{\alpha,-c}$ when $n = 2k+1$, and are $T_{\alpha,c}$ and $T_{\alpha^\vee,-c}$ when $n = 2k+2$, where $\alpha^\vee = (-n_1, n_2, n_3, \ldots, n_k)$ for $\alpha = (n_1, n_2, \ldots, n_k)$.

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<u>Appendix</u>. Diagram of symmetries in the space of elementary representations of Lorentz groups

Let $G = SO_0(n-1, 1)$ $(n \geqslant 3)$, and $\tilde{G}$ its two-fold covering group. Then the quasi-simple irreducible representations of $\tilde{G}$ are classified in [II]. Let $\tilde{M}$ be the subgroup of $\tilde{G}$ correspon-ding to $M \subset G$, and $\tilde{M}^{\wedge}$ the set of equivalent classes of irreduci-ble representations of $\tilde{M}$. For $\alpha \in \tilde{M}^{\wedge}$ and $c \in \mathbb{C}$, we also define a representation $T_{\alpha,c}$ of $\tilde{G}$ canonically as in [B] above. These representations are called elementary representations of G. When $T_{\alpha,c}$ is irreducible, it is denoted by $\mathcal{D}_{(\alpha;c)}$ in [II], and the notations $D^{\pm}_{(\alpha;p)}$, $D^{j}_{(\alpha;p)}$ and $\mathfrak{S}_{\mu}$ in [II] are also employed here.

The intertwining relations (or symmetries) between $T_{\alpha,c}$'s are devided into two types: those corresponding to the elements of the restricted Weyl group (type I), and others called discrete symmet-ries (type II). (Note that in this case the resricted Weyl group is of order 2.)

When $T_{\alpha,c}$ is reducible, we employ the following notation. Let U be the restriction of $T_{\alpha,c}$ on an invariant subspace H' of H_α, and V the representation on H_α/H' induced from $T_{\alpha,c}$, then we denote this fact by $T_{\alpha,c} = V \rightarrow U$. Let π_U and π_V be the characters of U and V respectively, then

(1) $\quad \pi_U + \pi_V =$ the character of $T_{\alpha,c}$.

(The character of $T_{\alpha,c}$ is given by the same formula as that for $\mathcal{D}_{(\alpha,c)}$ in [III] evenwhen it is not irreducible.)

Since the result is parallel for G and $\tilde{G}$, we present it at the same time. Here we give (i) the structure of $T_{\alpha,c}$ when it is reducible, and (ii) the symmetries between $T_{\alpha,c}$'s. For convenience, we divide n into two cases: (I) $n = 2k + 2$ $(k \geqslant 1)$, and (II) $n = 2k + 3$ $(k \geqslant 0)$.

Notation. Every element $\alpha \in \tilde{M}^\wedge$ is represented by a dominant integral form on a fixed Cartan subalgebra $\underline{b}$ of the Lie algebra of M. Denote by ρ half the sum of all positive roots (with respect to $\underline{b}$). Then $T_{\alpha,c}$ is represented also by $(\alpha + \rho; c)$ as a point in the space of all elementary representations. When $T_{\alpha,c}$ is reducible, the point $(\alpha + \rho; c)$ is called singular.

Case I. $n = 2k + 2$ $(k \geqslant 1)$.

In this case, $M = SO(2k)$, and $\alpha \in \tilde{M}^\wedge$ is canonically represented as

(I-1) $$\alpha = (n_1, n_2, \ldots, n_k),$$

where $n_1, n_2, \ldots, n_k$ are integers or half-integers at the same time such that

(I-2) $$|n_1| \leqslant n_2 \leqslant n_3 \leqslant \cdots \leqslant n_k .$$

(A number is called a half-integer if it is a half of an odd integer.) Then

(I-3) $$\rho = (0, 1, 2, \ldots, k-1).$$

We put

(I-4) $$\alpha + \rho = (\ell_1, \ell_2, \ldots, \ell_k) = (n_1, n_2 + 1, n_3 + 2, \ldots, n_k + (k-1)).$$

(i) Structure of $T_{\alpha,c}$ at singular points

The representation $T_{\alpha,c}$ is reducible if and only if
(α) c is of the same parity with ℓ_j (i.e., $c - \ell_j$ is an integer), and (β) $|c|$ is bigger than $|\ell_1|$, and different from ℓ_2, ℓ_3, ..., ℓ_k.

Since $T_{\alpha^\vee,-c}$ with $\alpha^\vee = (-n_1, n_2, n_3, \ldots, n_k)$ is the contragradient representation of $T_{\alpha,c}$, it is sufficient for us to treat $c \geqslant 0$. The structure of $T_{\alpha,c}$'s is summalized in Table I-1. (The number $c \geqslant 0$ is assumed there to satisfy (α), (β).)

(ii) Symmetries in the space of elementary representations

The representation $T_{\alpha,c}$ and its contragradient one $T_{\alpha^\vee,-c}$ are connected by the intertwining operators of type I: denoting $T_{\alpha,c}$ by $(\alpha + \rho; c)$, this is illustrated as

$$(\alpha^\vee + \rho; -c) \rightleftarrows (\alpha + \rho ; c).$$

When the point $(\alpha + \rho; c)$ is not singular, this gives an isomorphism by an invertible bounded operator, and when the point is singular, this gives a homomorphism onto the non-trivial invariant subspace.

Let $(m_1, m_2, \ldots, m_{k+1})$ be such that m_j's are integers or half-integers at the same time and that

$$\text{(I-5)} \qquad 0 \leqslant |m_1| < m_2 < \cdots < m_{k+1} .$$

Then $(\alpha + \rho; c) = (m_1, m_2, \ldots, \widehat{m_j}, \ldots, m_{k+1}; m_j)$ and its contragradient $(\alpha^\vee + \rho; -c) = (-m_1, m_2, \ldots, \widehat{m_j}, \ldots, m_{k+1}; -m_j)$ are singular points for any $j \geqslant 2$. (Here $\widehat{m_j}$ indicates that m_j is absent.) Conversely any singular point appears in this fashion.

The above 2k-points are connected by the symmetries as is indicated in Table I-2.

Notation. In Tables I-2, II-2, the arrows $\longrightarrow$ and $\Longrightarrow$ denote the intertwining maps of type I and of type II respectively, which are never ismorphism at singular points, and the arrow $\dashrightarrow$ denotes non-trivial homomorphism obtained by an iteration of these maps.

Note. The existence of intertwining operators illustrated by the arrows in Tables I-2 and II-2 is affirmed as follows (especially for $\Longrightarrow$). First these operators can be determined uniquely (up to constant multiples) with respect to the basis $\xi'(\lambda)$ from the infinitesimal stand point by using the representations $S_{\alpha,c}$ of $\underline{g}$. Then we see these operators, defined only for the vectors $\xi'(\lambda)$'s, can be extended to bounded operators on the respective Hilbert spaces H_α's. (Taking into account the relation between $S_{\alpha,i\rho}$ and $T'_{\alpha,i\rho}$ in page 7, we may consider $\xi'(\lambda)$'s are orthonormal.)

References. We give here some references for Appendix, not only for Case I but also for Case II.

A. M. Gavrilik, A. U. Klimyk: Analysis of the representations of the Lorentz and Euclidean groups of n-th order, preprint, 1975, Institute for Theoretical Physics, Kiev.

A. U. Klimyk, A. G. Gavrilik: The representations of the groups U(n, 1) and $SO_0(n, 1)$, preprint, 1976, the same Institute.

Table I-1. Structure of $T_{\alpha,c}$ (Case $n = 2k + 2$)

1) $\ell_k < c$ (put $p = c - k$, then $n_k \leqslant p$):

$$T_{\alpha,c} = D^{k-1}_{(\alpha';p')} \longrightarrow \mathfrak{S}_\mu$$

where $\mu = (n_1, n_2, \ldots, n_k, p)$, and

$$\alpha' = (n_1, n_2, \ldots, n_{k-1}, p+1), \quad p' = n_k .$$

2) $\ell_{k-1} < c < \ell_k$ (put $p = c - (k - 1)$, then $n_{k-1} \leqslant p < n_k$):

$$T_{\alpha,c} = D^{k-2}_{(\alpha';p')} \longrightarrow D^{k-1}_{(\alpha;p)} ,$$

where $\alpha' = (n_1, n_2, \ldots, n_{k-2}, p+1, n_k)$, $p' = n_{k-1}$.

3) $\ell_{k-2} < c < \ell_{k-1}$ (put $p = c - (k - 2)$, then $n_{k-2} \leqslant p < n_{k-1}$):

$$T_{\alpha,c} = D^{k-3}_{(\alpha';p')} \longrightarrow D^{k-2}_{(\alpha;p)} ,$$

where $\alpha' = (n_1, n_2, \ldots, n_{k-3}, p+1, n_{k-1}, n_k)$, $p' = n_{k-2}$.

4) In general for $2 \leqslant j \leqslant k-1$, $\ell_j < c < \ell_{j+1}$ (put $p = c - j$, then $n_j \leqslant p < n_{j+1}$):

$$T_{\alpha,c} = D^{j-1}_{(\alpha';p')} \longrightarrow D^{j}_{(\alpha;p)} ,$$

where $\alpha' = (n_1, n_2, \ldots, n_{j-1}, p+1, n_{j+1}, \ldots, n_k)$, $p' = n_j$.

5) $|\ell_1| < c < \ell_2$ (put $p = c - 1$, then $|n_1| \leqslant p < n_2$):

$$T_{\alpha,c} = \mathcal{D}_{(\alpha';c')} \longrightarrow D^{1}_{(\alpha;p)} ,$$

where $\alpha' = (p+1, n_2, n_3, \ldots, n_k)$, $c' = n_1$.

Note. When $k = 1$, the relation in 1) turns out to the following:

for $|\ell_1| < c$ (put $p = c - 1$, then $|n_1| \leqslant p$),

$$T_{n_1,c} = \mathcal{D}_{(c;n_1)} \longrightarrow \mathfrak{S}_{(n_1,p)} .$$

Table I-2. Symmetries (Case $n = 2k + 2$)

Structure factor space invariant subspace $V \to U$	$(\alpha^\vee + \rho;\ -c)$		$(\alpha + \rho;\ c)$	Structure factor space invariant subspace $V \to U$
$\mathcal{C} \to D^{k-1}$	$(-m_1, m_2, \ldots, m_k; m_{k+1})$	$\rightleftarrows$	$(m_1, m_2, \ldots, m_k; m_{k+1})$	$D^{k-1} \to \mathcal{C}$
	$\Uparrow$		$\Downarrow$	
$D^{k-1} \to D^{k-2}$	$(-m_1, m_2, \ldots, m_{k-1}, m_{k+1}; -m_k)$	$\rightleftarrows$	$(m_1, m_2, \ldots, m_{k-1}, m_{k+1}; m_k)$	$D^{k-2} \to D^{k-1}$
	$\Uparrow$		$\Downarrow$	
....	...... $\vdots$		...... $\vdots$	......
	$\Uparrow$		$\Downarrow$	
$D^{j-1} \to D^{j-2}$	$(-m_1, m_2, \ldots, \hat{m}_j, \ldots, m_{k+1}; -m_j)$	$\rightleftarrows$	$(m_1, m_2, \ldots, \hat{m}_j, \ldots, m_{k+1}; m_j)$	$D^{j-2} \to D^{j-1}$
	$\Uparrow$		$\Downarrow$	
....	...... $\vdots$		.... $\vdots$	......
	$\Uparrow$		$\Downarrow$	
$D^1 \to \mathcal{D}$	$(-m_1, \hat{m}_2, m_3, \ldots, m_{k+1}; -m_2)$	$\rightleftarrows$	$(m_1, \hat{m}_2, m_3, \ldots, m_{k+1}; m_2)$	$\mathcal{D} \to D^1$
	$\Uparrow$		$\Downarrow$	
$\mathcal{D}$ (irred.)	$(\hat{m}_1, -m_2, m_3, \ldots, m_{k+1}; -m_1)$	$\rightleftarrows$	$(\hat{m}_1, m_2, \ldots, m_{k+1}; m_1)$	$\mathcal{D}$ (irred.)

The arrows $\rightleftarrows$ and $\Rightarrow$ denotes the intertwinig maps of type I and of type II respectively, and the arrow $\dashrightarrow$ denotes non trivial homomorphism obtained by an iteration of two maps.

Case II. $n = 2k + 3$ $(k \geqslant 0)$.

In this case, $M = SO(2k+1)$, and $\alpha \in \widetilde{M}^{\wedge}$ is canonically represented as

(II-1) $\alpha = (n_1 , n_2 , \dots, n_k)$,

where $n_1 , n_2 , \dots, n_k$ are integers or half-integers at the same time such that

(II-2) $0 \leqslant n_1 \leqslant n_2 \leqslant \dots \leqslant n_k$.

Then

(II-3) $\rho = (1/2, 3/2, \dots, (2k-1)/2)$,

(II-4) $\alpha + \rho = (\ell_1 , \ell_2 , \dots, \ell_k)$
$= (n_1 + 1/2, n_2 + 3/2, \dots, n_k + (2k-1)/2)$.

(i) Structure of $T_{\alpha,c}$ at singular points

The representation $T_{\alpha,c}$ is reducible if and only if (α) c is of the same parity with ℓ_j (i.e., $c - \ell_j$ is an integer), and (β) c is different from $\pm\ell_1 , \pm\ell_2 , \dots, \pm\ell_k$.

Since $T_{\alpha,-c}$ is contragradient to $T_{\alpha,c}$, it is sufficient to treat $c \geqslant 0$. The structure of $T_{\alpha,c}$'s is summalized in Table II-1. (Then number $c \geqslant 0$ is assumed there to satisfy (α), (β).)

(ii) Symmetries in the space of elementary representations

The representation $T_{\alpha,c}$ and its contragradient one $T_{\alpha,-c}$

are connected by the intertwining operators of type I:

$$(\alpha + \rho; -c) \rightleftarrows (\alpha + \rho; c).$$

The analogous facts hold also in this case as in the Case I: $n = 2k + 2$. (Note that the points $(\alpha + \rho; c)$ with $c = 0$, that is, $(\alpha + \rho; -c) = (\alpha + \rho; c)$, are singular when ℓ_j's are integers in Case II.)

Let $(m_1, m_2, \ldots, m_{k+1})$ be such that m_j's are integers or half-integers at the same time and that

$$\text{(II-5)} \qquad 0 < m_1 < m_2 < \cdots < m_{k+1} .$$

Then $(\alpha + \rho; c) = (m_1, m_2, \ldots, \widehat{m_j}, \ldots, m_{k+1}; m_j)$ and its contragradient one $(\alpha + \rho; -c) = (m_1, m_2, \ldots, \widehat{m_j}, \ldots, m_{k+1}; -m_j)$ are singular points for any $j \geqslant 1$. Conversely any singular point appears in this fashion except the points $(\alpha + \rho; c)$ with $c = 0$. The above $2(k+1)$-points are connected by the symmetries as is indicated in Table II-2.

The points $(\alpha + \rho; c)$ with $c = 0$ and ℓ_j integers, are all singular, but any such $(\alpha + \rho; 0)$ is not connected with another point by symmetries. In this sense, these points may be called as "isolated" singular points. They correspond to $(m_1, m_2, \ldots, m_{k+1})$ analogous as above such that

$$\text{(II-5')} \qquad 0 = m_1 < m_2 < \cdots < m_{k+1} .$$

<u>Note</u>. The irreducible components $D^{\pm}_{(\alpha;1/2)}$ of $T_{\alpha,0}$ are the limits of the discrete series representations similarly as for $SO_0(2,1)$, in the case where n_j's are half-integers, or equivalently, ℓ_j's are integers.

Table II-1. Structure of $T_{\alpha,c}$ (Case $n = 2k + 3$)

1) $\ell_k < c$ (put $p = c - (2k+1)/2$, then $n_k \leqslant p$):

$$T_{\alpha,c} = D^k_{(\alpha';p')} \longrightarrow \mathfrak{S}_\mu$$

where $\mu = (n_1, n_2, \ldots, n_k, p)$, and
$\alpha' = (n_1, n_2, \ldots, n_{k-1}, p+1)$, $p' = n_k$.

2) $\ell_{k-1} < c < \ell_k$ (put $p = c - (2k-1)/2$, then $n_{k-1} \leqslant p < n_k$):

$$T_{\alpha,c} = D^{k-1}_{(\alpha';p')} \longrightarrow D^k_{(\alpha;p)} ,$$

where $\alpha' = (n_1, n_2, \ldots, n_{k-2}, p+1, n_k)$, $p' = n_{k-1}$.

3) In general for $2 \leqslant j \leqslant k$, $\ell_{j-1} < c < \ell_j$ (put $p = c - (2j-1)/2$, then $n_{j-1} \leqslant p < n_j$):

$$T_{\alpha,c} = D^{j-1}_{(\alpha';p')} \longrightarrow D^j_{(\alpha;p)} ,$$

where $\alpha' = (n_1, \ldots, n_{j-2}, p+1, n_j, \ldots, n_k)$, $p' = n_{j-1}$.

4) $0 < c < \ell_1$ (put $p = c - 1/2$, then $0 \leqslant p < n_1$):

$$T_{\alpha,c} = (D^+_{(\alpha;p')} + D^-_{(\alpha;p')}) \longrightarrow D^1_{(\alpha;p)} ,$$

where $p' = p + 1 = c + 1/2 \geqslant 1$.

5) $c = 0$ (with ℓ_j integers):

$$T_{\alpha,0} = D^+_{(\alpha;1/2)} + D^-_{(\alpha;1/2)} .$$

Note. When $k = 0$, $\tilde{M}$ is the center of $\tilde{G}$ and of order 2. We express $\alpha \in \tilde{M}^\wedge$ as $\alpha = 0$ or $1/2$ according as it is trivial or not. Then the relation 1) turns out to the following: let $c > 0$ be a half-integer or an integer according as $\alpha = 0$ or $1/2$, and put $\mu = c - 1/2$, then

$$T_{\alpha,c} = (D^+_{c+1/2} \oplus D^-_{c+1/2}) \longrightarrow \mathfrak{S}_\mu .$$

Table II-2. *Symmetries* (*Case* $n = 2k + 3$)

Structure (factor space → invariant subspace) $V \to U$	$(\alpha + \rho;\ -c)$		$(\alpha + \rho;\ c)$	Structure (factor space → invariant subspace) $V \to U$
$\mathfrak{S} \to D^k$	$(m_1, m_2, \ldots, m_k; -m_{k+1})$	$\rightleftarrows$	$(m_1, m_2, \ldots, m_k; m_{k+1})$	$D^k \to \mathfrak{S}$
	$\Uparrow$		$\Downarrow$	
$D^k \to D^{k-1}$	$(m_1, m_2, \ldots, m_{k-1}, m_{k+1}; -m_k)$	$\rightleftarrows$	$(m_1, m_2, \ldots, m_{k-1}, m_{k+1}; m_k)$	$D^{k-1} \to D^k$
	$\Uparrow$		$\Downarrow$	
......	....		....	......
	$\Uparrow$		$\Downarrow$	
$D^j \to D^{j-1}$	$(m_1, m_2, \ldots, \hat{m}_j, \ldots, m_{k+1}; -m_j)$	$\rightleftarrows$	$(m_1, m_2, \ldots, \hat{m}_j, \ldots, m_{k+1}; m_j)$	$D^{j-1} \to D^j$
	$\Uparrow$		$\Downarrow$	
......	....		....	......
	$\Uparrow$		$\Downarrow$	
$D^2 \to D^1$	$(m_1, \hat{m}_2, \ldots, m_{k+1}; -m_2)$	$\rightleftarrows$	$(m_1, \hat{m}_2, \ldots, m_{k+1}; m_2)$	$D^1 \to D^2$
	$\Uparrow$		$\Downarrow$	
$D^+ \oplus D^- \to D^1$	$(\hat{m}_1, m_2, \ldots, m_{k+1}; -m_1)$	$\rightleftarrows$	$(\hat{m}_1, m_2, \ldots, m_{k+1}; m_1)$	$D^+ \oplus D^- \to D^1$

Here $0 < m_1 < m_2 < \ldots < m_{k+1}$ are all integers or half-integers at the same time. Besides these singular points, there exist "isolated" singular points $(\ell_1, \ell_2, \ldots, \ell_k; 0)$, where $0 < \ell_1 < \ell_2 < \ldots < \ell_k$ are integers.